MATH 2040 A Tutorial Week 10

§6.3 The adjoint

8. V fin. dim. v.s. with inner product. T a linear operator on V. Prove that if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Solution: $T^*(T^{-1})^* = (T^{-1}T)^* = I^* = I$

10. T a linear operator on an inner product space V. Prove that $\|T(x)\| = \|x\|$ for all $x \in V$ iff $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$.
(Hint: Ex 20 of 6.1)

Solution: "$\Leftarrow$" take $y = x$

"$\Rightarrow$" By Ex 20 of 6.1, we have when $F = \mathbb{C}$

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2 = \frac{1}{4}\sum_{i=1}^{4} i^k \|T(x + i^k y)\|^2$$

$$= \frac{1}{4}\sum_{k=1}^{4} i^k \|T(x) + i^k T(y)\|^2 = \langle T(x), T(y)\rangle$$

$F = \mathbb{R}$ is similar.

12. V an inner product space. T a linear operator on V. Prove

a) $R(T^*)^\perp = N(T)$

b) If V is fin. dim., then $R(T^*) = N(T)^\perp$. (Hint: Use Ex 13 (c) of 6.2)

Solution: a) If $x \in R(T^*)^\perp$, then $0 = \langle x, T^*(y)\rangle = \langle T(x), y\rangle \quad \forall y$.

$\Rightarrow T(x) = 0 \Rightarrow x \in N(T)$.

If $x \in N(T)$, then $\langle x, T^*(y)\rangle = \langle T(x), y\rangle = 0 \quad \forall y$

$\Rightarrow x \in R(T^*)^\perp$.

b) By Ex 13 (c) of 6.2,

$$N(T)^\perp = (R(T^*)^\perp)^\perp = R(T^*)$$

§6.3 Normal and Self-adjoint operator

7. T a linear operator on an inner product space V. W a T-invariant space of V.

Prove

a) If T is self-adjoint, then T_W is self-adjoint.

b) $W^\perp$ is T^*-invariant

c) If W is both T and T^* invariant, then $(T_W)^* = (T^*)_W$

d) If W is both T and T^* invariant and T is normal, then T_W is normal

Solution: a) $\langle x, (T_W)^*(y) \rangle = \langle T_W(x), y \rangle = \langle T(x), y \rangle = \langle T^*(x), y \rangle$

$= \langle x, T(y) \rangle = \langle x, T_W(y) \rangle \quad \forall x, y \in W$

b) $y \in W^\perp$, $\langle x, T^* y \rangle = \langle Tx, y \rangle = 0 \quad \forall x \in W$

(Since $T(x) \in W$ W is T-invariant)

c) $\langle x, (T_W)^*(y) \rangle = \langle T_W x, y \rangle = \langle T(x), y \rangle = \langle x, T^* y \rangle$

$= \langle x, (T^*)_W(y) \rangle$

d) T normal $\Rightarrow T^*T = TT^*$.

W is both T and T^* invariant, $(T_W)^* = (T^*)_W$ by previous argument.

$\Rightarrow T_W (T_W)^* = T_W (T^*)_W = (T^*)_W T_W = (T_W)^* T_W$

9. T a normal operator on a fin. dim. inner product space V. Prove that

$N(T) = N(T^*)$ and $R(T) = R(T^*)$

Solution: By Thm 6.15(a), $T(x) = 0$ iff $T^*(x) = 0$. $\Rightarrow N(T) = N(T^*)$

Ex 12 of 6.3 $\Rightarrow R(T) = N(T^*)^\perp = N(T)^\perp = R(T^*)$

12. Let T be a normal ~~space~~ operator on a fin dim real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigenvectors of ~~T~~ T. Hence prove that T is self-adjoint.

Solution: Characteristic polynomial splits $\Rightarrow$ (By Schur's theorem/decomposition) $\exists$ an orthonormal basis β s.t. $[T]_\beta$ is upper triangular.

$\beta := \{v_1 \cdots v_n\}$.

v_1 is an eigenvector.

$t :=$ the maximum integer s.t. $v_1 \cdots v_t$ are all eigenvectors w.r.t. eigenvalues λ_i.

If $t=n$, we are done.

If not. $[T]_\beta =: \{A_{i,j}\}$. $\Rightarrow T(v_{t+1}) = \sum_{i=1}^{t+1} A_{i,t+1} v_i$.

$A_{i,t+1} = \langle T(v_{t+1}), v_i \rangle = \langle v_{t+1}, T^*(v_i) \rangle = \langle v_{i+1}, \overline{\lambda_i} v_i \rangle = 0$.

$\Rightarrow v_{t+1}$ is also an eigenvector.

$\Rightarrow v_i$'s are eigenvectors of T.

$\Rightarrow$ (Thm 6.17) T is self-adjoint.

13. An $n \times n$ real matrix A is Gramian if $\exists$ a real square matrix B s.t. $A = B^t B$. Prove that A is Gramian iff A is symmetric and all of its eigenvalues are non-negative.

Proof: A is Gramian $\Rightarrow$ symmetric

λ eigenvalue with unit eigenvector x. Then $Ax = \lambda x$

$\lambda = \langle Ax, x \rangle = \langle Bx, Bx \rangle \geq 0$

Conversely, if A is symmetric, then L_A is a self-adjoint operator. $\exists$ an orthonormal basis β s.t. $[L_A]_\beta$ is diagonal ($= \mathrm{diag}\{\cdots \lambda_i \cdots\}$)

$D := \mathrm{diag}\{\cdots \sqrt{\lambda_i} \cdots\}$ $\Rightarrow D^2 = [L_A]_\beta \Rightarrow A = ([I]^\alpha_\beta D)(D[I]^\beta_\alpha)$.

(α - standard basis). $[I]^\alpha_\beta = ([I]^\beta_\alpha)^t \Rightarrow B := D[I]^\beta_\alpha \quad A = B^t B$